

# K-Essential Phantom Energy: Revisited

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**Abstract** We generalize some of those results which are closely related to the canonical kinetic term in  $k$ -essence formalism by further tuning the parameter ( $\beta$ ). The scale factor  $a(t)$  could be negative and decreasing within a specific range of  $\beta$  ( $\equiv -1/\omega$ ,  $\omega$ : the equation-of-state parameter) during the initial evolutionary period.

**Keywords** Physics of the early universe · Cosmology of theories beyond the Standard Model · Dark energy theory · Inflation

## 1 Introduction

The standard (hot) big bang (SBB) theory is an extremely successful one, and has been around for over 60 years, since Gamow originally proposed it [1–7]. Remarkably, for such a simple idea, it provides us with an understanding of many of the basic features of our Universe. All that you require in the cooking pot, are initial conditions of an expanding scale factor, gravity, plus the standard particle physics we are used to, to provide the matter in the Universe. However, as mentioned the hot big bang theory can successfully proceed only if the initial conditions are very carefully chosen, and even then it only really works at temperatures low enough, so that the underlying physics can be well understood. The very early Universe is out of bounds, yet there is a hope that accurate observations of the present state of the Universe may highlight the types of process occurring during these early stages, providing an insight on the nature of physical laws at energies which it would be inconceivable to explore by other means. Another unresolved issue is the cause of the apparent acceleration of the Universe today, as seen through the distribution of distant Type Ia Supernovae.

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The physical properties of vacuum phantom energy are rather weird, as they include violation of the dominant-energy condition,  $P + \rho < 0$  (with the equation of state:  $P = \omega\rho$  [8–14],  $\omega < -1$ ;  $P$  is the homogeneous dark energy pressure and  $\rho$  is the energy density), naïve superluminal sound speed and increasing vacuum-energy density. The latter property ultimately leads to the emergence of a singularity usually referred to as big rip in a finite time in future where both the scale factor and the vacuum-energy density blow up [15–20]. For instance, it was shown that a cosmological model with a singular big rip at an arbitrary finite time in the future can be also obtained when the scalar field satisfies equivalent phantom-energy conditions in the case that it is equipped with non-canonical kinetic energy [9–14] for models restricted by a Lagrangian of the form  $\mathcal{L} = K(\phi)q(y)$  [15]. It was announced that, one can play with the arbitrary values of the prefactor  $a_0$  for the scale factor expression in (10) and those unboundedly small positive values of  $t$  which satisfy the observational constraint [15] (note that in this model  $\beta = -1/\omega_\phi = -1/\omega$ ) and the currently observed cosmic acceleration rate [21–23] to check that such a set of present observations [21–23] is compatible with unboundedly small positive values of  $t$  (the assumption is, if a quintessential scalar field  $\phi$  with constant equation of state  $P_\phi = \omega_\phi \rho_\phi = \omega \rho_\phi$  is considered, then phantom energy can be introduced by allowing violation of dominant energy condition,  $P_\phi + \rho_\phi < 0$ ). Note that, in general  $k$ -essence is defined as a scalar field with non-canonical kinetic energy, but usually the models are restricted to the above Lagrangian form ( $\mathcal{L}$ ) [24]. Normally we use units where  $8\pi G/3 = 1$  [9–14].

It has been proposed that, from the perfect-fluid analogy, we have for the pressure and energy density of a generic  $k$ -essence scalar field  $\phi$  [24]

$$P_\phi(y) = K(\phi)g(y)/y, \quad \rho_\phi(y) = -K(\phi)g'(y),$$

where

$$g(y) = By^\beta,$$

with  $B$  and  $\beta$  being given constants such that  $B > 0$  and  $0 < \beta < 1$  and the prime means derivative with respect to  $y$ . Thus, the equation-of-state parameter reads

$$\omega = \frac{-g(y)}{yg'(y)}.$$

We set next the general form of the function  $g(y)$  when we consider a phantom-energy  $k$ -essence field; i.e., when we introduce the following two phantom energy conditions:  $K(\phi) < 0$  and  $P_\phi(y) + \rho_\phi(y) < 0$ . The weak energy condition  $\rho_\phi(y) > 0$  is presumed to be valid here so that  $g'(y) > 0$ . From the second phantom-energy condition, we then deduce that  $g(y) > yg'(y)$  (and  $g(y) > 0$ ). Therefore the function  $g(y)$  should be an increasing concave function, that is we must also set  $d^2g(y)/dy^2 < 0$ .

Following [15], we then specialize in the case of a spatially flat Friedmann–Robertson–Walker spacetime with line element  $ds^2 = -dt^2 + a(t)^2dr^2$ , in which  $a(t)$  is the scale factor. In the case of a universe dominated by a  $k$ -essence phantom vacuum energy, the Einstein field equations are then [9–15]

$$3H^2 = \rho_\phi(y), \quad 2\dot{H} + \rho_\phi(y) + P_\phi(y) = 0,$$

with  $H = \dot{a}/a$ , the overhead dot meaning time derivative.

By combining the two expressions above and using the equation of state he can obtain for the function  $g(y)$  as given above

$$3H^2 = \frac{2\dot{H}\beta}{1-\beta}, \tag{1}$$

and then the solutions (for his spatially flat case)

$$a(t) = \frac{a_0}{(t-t_*)^{2\beta/[3(1-\beta)]}}, \quad 0 < \beta < 1, \tag{2}$$

where  $t_*$  is an arbitrary and positive constant (this positive  $t_*$  solution family does represent an accelerating universe if  $a(t)$  increases once  $t$  increases [15]). The behavior of  $a(t)$  has been presented in Fig. 1 (II) therein [15] and some remarks were made [15]: of quite greater interest is the choice  $t_* > 0$  for which the universe (Fig. 1 (II), solid line [15]) will first expand to reach a big rip singularity at the arbitrary time  $t = t_*$  in the future, to thereafter steadily collapse to zero at infinity; that is it matches the behavior expected for current quintessence models with  $\omega < -1$ . The potentially dramatic difference is that whereas in quintessence models the time at which the big rip will occur depends nearly inversely on the absolute value of the state equation parameter, in the present k-essence model the time  $t_*$  is a rather arbitrary parameter [15].

The present author, however, based on similar derivations ( $1/y^2 = \dot{\phi}^2/2$  [9–15]) could obtain (10) the same as that in [15] but different behavior of  $a(t)$  for  $t < t_*$  region with  $\beta = 1/2223, 0.60$  and  $297/299$  (compared to the (II) of Fig. 1 in [15] which is for  $0 < \beta < 1$ ). Our testing results with  $\beta = 1/2223, 0.60, 0.85, 297/299$ , respectively (select  $t_* = 2.0$ ), however, can clearly illustrate that the curves of  $\beta = 1/2223, 0.60, 297/299$  (in  $t < t_*$  region) are different from that claimed in Fig. 1 (II) in [15]. Under these situations, values of  $a(t)$  (which are negative, if we define

$$B_p = \frac{2\beta}{3(1-\beta)} = \frac{-2/\omega}{3(1+\frac{1}{\omega})} = \frac{-2}{3(\omega+1)}, \tag{3}$$

we have  $B_p = 1/3333, 1, 99$  for  $\beta = 1/2223, 0.60, 297/299$ , respectively; then (the scale factor)

$$a(t) = \frac{a_0}{[(t-t_*)^{B_p}]_{B_p=1/3333, 1, 99}} \tag{4}$$

should be negative for  $t < t_*$ !) firstly decrease as (time)  $t$  increases until  $t \rightarrow t_*$ . Here,  $a_0$  is an arbitrary integration constant.

To be specific, considering  $\beta = 0.60$  and  $297/299$  (or  $\omega = -5/3$  and  $-299/297$ ), of which the role of dark energy can be played by physical fields with positive energy and negative pressure which violates the strong energy condition  $\rho + 3P > 0$  ( $\omega > -1/3$ ), we thus obtain

$$a(t) = a_0(t-t_*)^{-1}, \quad a(t) = \frac{a_0}{(t-t_*)^{99}} \tag{5}$$

and there is no doubt that  $a(t)$  should be negative once  $t < t_*$ . These cases correspond to the earlier collapsing ones (but not collapse to zero) instead of expansion.

The  $\beta = 0.85$  curve (in  $t < t_*$  region), however, resembles that presented in Fig. 1 (II) of [15]. It means, for some cases of  $\beta$ , the universe will not first expand to reach a big rip singularity at the arbitrary time  $t = t_*$  in the future as claimed in [15]. Cases of  $a(t)$  being

negative (in  $t < t_*$  region) should be interpreted in another way (at lease for cases of  $\beta = 1/2223, 0.60, 297/299$  where the power:  $B_p$  is, at least, an odd integer or the denominator of  $B_p$  is an odd integer while the numerator of  $B_p$  is normalized to be 1)!

In fact, we can set

$$B_p = \frac{n}{m}, \quad n, m \neq 0, \tag{6}$$

where  $n, m$  are positive integers. We already demonstrated cases of  $B_p = 1/3333, 1,$  and  $99$  which correspond to  $(n, m) = (1, 3333), (1, 1), (99, 1)$  or  $\beta = 1/2223, 3/5, 297/299$  (or  $\omega = -2223, -5/3, -299/297$ ). To let  $B_p$  be a positive odd integer, we can select  $m = 1$  and  $B_p = n$ . It leads to

$$\beta = \frac{3n}{3n + 2} = \frac{3B_p}{2 + 3B_p} = \frac{-1}{\omega}. \tag{7}$$

Under this choice,  $a(t)$  will decrease (for  $t < t_*$ ) following

$$a(t) = \frac{a_0}{(t - t_*)^{B_p}} = \frac{a_0}{(t - t_*)^n} \tag{8}$$

since  $n$  is an odd integer ( $t - t_* < 0$ )! For example, with  $m = 1, n = 101$ , we have decreasing  $a(t)$  for  $\beta = 303/305$ !

On the other hand, once we choose  $n = 1$ , together with  $m$  is a positive integer, it then sets  $B_p = 1/m$  and

$$\beta = \frac{3}{2m + 3} = \frac{-1}{\omega}. \tag{9}$$

With this, as we have shown above ( $n = 1, m = 3$ ),  $a(t)$  will decrease for  $t < t_*$  following

$$a(t) = \frac{a_0}{(t - t_*)^{B_p}} = \frac{a_0}{(t - t_*)^{1/m}} = \frac{a_0}{\sqrt[m]{t - t_*}}. \tag{10}$$

One another example, for  $\beta \sim 0$ , is  $m = 1001$  ( $B_p = 1/1001$ ) or  $\beta = 3/2005$ , which leads to

$$a(t) = \frac{a_0}{\sqrt[1001]{t - t_*}}.$$

To make a brief summary, for those cases considered in (7) and (9), we have earlier collapsing cases ( $a(t)$  decreasing once  $t < t_*$ ; a negative divergence!). However, as we have illustrated above, cases which don't belong to the restriction of (7) or (9), say,  $n$  is an even integer ( $m$  is still an odd integer), e.g.,  $\beta = 0.85 = 17/20$  gives  $n = 34$  and  $m = 9$  or  $B_p = 34/9$ , will match the results claimed in [15]: the universe will first expand to reach a big rip singularity at the arbitrary time  $t = t_*$ ! At least, our results can also be extended to the cases considered in [8].

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